Woven Braids and their Closures

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ABSTRACT

A special class of braids, called woven, is introduced and it is shown qcxthat every conjugation class of the braid group contains woven braids. In consequence tame links can be presented as plats and closures of woven braids. Restricting on knots we get the 'woven version' of the well-known theorem of Markov, giving moves that are capable of producing all woven braids with equivalent closures.

As corollary we obtain that a link in which each component is dyed with at least two different colors can be projected on a plane without crossing strands of the same color.

Finally, a table of all minimal woven braids for the 84 prime knots with at most nine crossings is appended. The average word length and the average number of entries per knot type turn out to be surprisingly small.

Keywords: Braids, plats, conjugation, algebraic link problem.

1. Introduction and Results

1.1. Intention

The basic aim that led to these results was to look out for simple representatives of conjugation classes of the braid group $\mathcal{B}_n$. In other words: Find a family $\mathcal{W}_n$ of braids with $\forall\beta \in \mathcal{B}_n : \exists\gamma \in \mathcal{B}_n : \gamma^{-1}\beta\gamma \in \mathcal{W}_n$! It is desirable to make this family $\mathcal{W}_n \subset \mathcal{B}_n$ in some sense as 'small' as possible.

F. Garside gave a very good solution to this problem in [1], constructing exactly one representative per conjugation class, thereby solving the conjugacy problem in $\mathcal{B}_n$. Unfortunately the characterization of these special braids (called "normal forms") is not at all a trivial task (see theorem 2.7 of [2]).

In this paper, looking for an other solution of the above-mentioned problem, we are willing to sacrifice uniqueness for the sake of simplicity. The idea is to try to let $\mathcal{W}_n$ consist only of braids that can be presented with a maximum (compared to their conjugates) number of constant strings (single straight lines).

1.2. Woven braids

Given the usual generators $(\sigma_i)_{i\in\{1,\dots,n-1\}}$ of $\mathcal{B}_n$, letting the ith string pass over the $i+1$st, and k positive integers $n_1,\dots,n_k$ with $n = n_1+\cdots+n_k$, we make the following abbreviations: ($\hat{}$ means omission)

$$A_{ij} \;:=\; \sigma_{j-1}\cdots\ \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\ \sigma_{j-1}^{-1} \qquad (1\le i<j\le n) \tag{1.1}$$
$$\pi_{n_1n_2\dots n_k} \;:=\; \sigma_1\sigma_2\cdots\ \hat{\sigma}_{n_1}\cdots\ \hat{\sigma}_{n_1+n_2}\cdots\ \hat{\sigma}_{n_1+\cdots+n_{k-1}}\cdots\ \sigma_{n_1+\cdots+n_k-1} \tag{1.2}$$

The A_{ij} generate the pure braid group $\mathcal{P}_n := \langle (A_{ij})_{1\le i<j\le n}\rangle$, $\mathcal{P}_n^j := \langle (A_{ij})_{1\le i<j}\rangle$ is called *group of pure j-braids* and $\pi_{n_1n_2\dots n_k}$ is a permutation braid.

From the theory of braids (e.g. [2]) it is known that the $\mathcal{P}_n^j$ are free groups of rank $j-1$, that $\mathcal{P}_n$ is the semidirect product of $\mathcal{P}_n^2,\dots,\mathcal{P}_n^n$ and that every braid can be presented as product of a permutation braid and a pure braid. We will now focus on special combinations of pure and permutation braids.

Definition 1. A braid on $n = n_1+\cdots+n_k$ strings that can be presented in the form $\pi_{n_1\dots n_k}\beta_{n_1}\beta_{n_1+n_2}\cdots\ \beta_{n_1+\cdots+n_k}$ with each $\beta_i\in\mathcal{P}_n^i$ is called a *woven* braid (with k *components*) of *type* $(n_1,\dots,n_k)$. The set of all woven braids on n strings with k components will be called $\mathcal{W}_n^k$.

Pictures of woven braids of type (4), (5) and (3,1,2) are given in figure 1.[1]

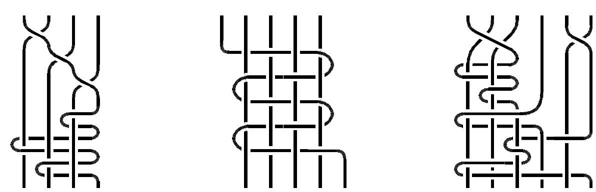

Fig. 1. Examples of woven braids.

We will prove in section 2 that $\mathcal{W}_n := \bigcup_{k=1}^n \mathcal{W}_n^k$ indeed has the desired property:

Theorem 1. *Every conjugation class of $\mathcal{B}_n$ contains at least one element of $\mathcal{W}_n$.*

1.3. Woven braids and links

For convenience, the closure of a woven braid of the above type is named *web of index n* and the plat formed out of a woven braid whose type is a k-tuple of *even* integers with sum $2n$ is called *woven plat of index n*.[2]

[1] The second example explains our terminology, for this is how typical outcomes of looms look like.
[2] The closure of a braid is obtained by identifying corresponding left and right endpoints; a plat is built by grouping the endpoints on each side into non-separating pairs, and then glueing the members of each pair together.

It is easily seen that both constructions yield a link with k components. We are interested only in tame links, and equivalence of link representatives shall be given by ambient isotopy. A link representative is called *multicolored* if each embedded circle is split up into a finite collection of arcs, and a color is attached to each arc, using at least two different colors per component. With these notations we can state simple consequences of theorem 1:

Corollary 1. *Let L be an arbitrary equivalence class of links, then:*
- (a) *L contains (infinitely many) webs and woven plats.*
- (b) *The minimal index of all webs in L equals the braid index of L.*
- (c) *The minimal index of all woven plats in L equals the bridge number of L.*
- (d) *For every multicolored representative l of L there exists a color preserving ambient isotopy taking l to $\tilde{l}$ and a projection of $\tilde{l}$ to a colored regular planar diagram, in which the two strands of every crossing have different colors.*

Part (a) of this corollary is just a sharper version of Alexander's theorem ([3]), and one might ask for a specialization of Markov's theorem ([4] and [2]) as well. The following subsection gives the answer for the $k = 1$ case.

1.4. Woven braids and knots

Definition 2. Let ∇ be the endomorphism of $\mathcal{P}_n$ given by

$$\nabla(A_{ij}) := \begin{cases} 1_{\mathcal{P}_n} & \text{for } i = 1 \\ A_{i-1,j-1} & \text{for } i \geq 2 \end{cases} . \tag{1.3}$$

To see that ∇ is well-defined, check out that it is compatible with the defining relations of $\mathcal{P}_n$ that can be found in [5]. Furthermore let

$$\mathcal{K}_n := \{ \; \beta \, \nabla(\beta) \, \nabla\big(\nabla(\beta)\big) \cdots \nabla^{n-2}(\beta) \mid \beta \in \mathcal{P}_n^n \; \} \tag{1.4}$$

and define the following moves for woven braids:

Type I move:	Replace $\omega \in \mathcal{W}_n^1$ by $\kappa^{-1}\omega\kappa$ for some $\kappa \in \mathcal{K}_n$.
Type II$^+$ move:	Replace $\omega \in \mathcal{W}_n^1$ by $\omega\sigma_n^{\pm 1}$.
Type II$^-$ move:	Replace $\omega\sigma_{n-1}^{\pm 1} \in \mathcal{W}_n^1$ by ω when $\omega \in \mathcal{W}_{n-1}^1$.

Remark. An explicit specification of the string number n is not necessary here, for the permutation of a braid in $\mathcal{W}_n^1$ is a cycle of length n.

Theorem 2.
- (a) *Application of a type I and II$^\pm$ move on a woven braid of type (n) yields a woven braid of type (n) and $(n \pm 1)$, respectively.*
- (b) *$\mathcal{K}_n$ is a subgroup of $\mathcal{P}_n$ operating on $\mathcal{W}_n^1$ by conjugation; the orbits are the intersections of the conjugation classes of $\mathcal{B}_n$ with $\mathcal{W}_n^1$.*
- (c) *Two webs represent the same knot, if and only if the corresponding woven braids are related via a finite sequence of type I, II$^\pm$ moves.*

2. Weaving Braids by Conjugation

If for some fixed $s \in \{2, \dots, n\}$ the pure braid β can be written as word of $\mathcal{P}_n$ without using letters A_{is} i.e. $\beta \in \langle (A_{ij})_{j \neq s} \rangle$, then we say string s is *free* in β.

Lemma 2.1. *For any $\beta \in \mathcal{P}_n$ and $s \in \{2, \dots, n\}$ there exist two unique pure s-braids ${}_s\beta, \beta_s \in \mathcal{P}_n^s$ such that string s is free in $({}_s\beta)^{-1}\beta$ and in $\beta(\beta_s)^{-1}$.*

Remark. $\beta_s = {}_s\beta = 1_{\mathcal{P}_n}$ if and only if string s is free in β.

Proof. Due to the structure of $\mathcal{P}_n$, being the semidirect product of the $\mathcal{P}_n^k$, pure i-braids commute with pure j-braids in the sense that for all $i < j$, $\mu \in \mathcal{P}_n^i, \nu \in \mathcal{P}_n^j$ there exists a $\tilde{\nu} \in \mathcal{P}_n^j$ such that $\nu\mu = \mu\tilde{\nu}$. Thus one can push all subwords $\in \mathcal{P}_n^s$ of a word for β to the beginning or to the end where they can be cancelled.
Uniqueness: If we have two such $\grave{\beta}_s, \acute{\beta}_s \in \mathcal{P}_n^s$ then string s has to be free in $\left(\beta\grave{\beta}_s^{-1}\right)^{-1}\beta\acute{\beta}_s^{-1} = \grave{\beta}_s\acute{\beta}_s^{-1}$ as well. This implies $\grave{\beta}_s = \acute{\beta}_s$ (analogously for ${}_s\beta$). □

Lemma 2.2. *For any $\beta \in \mathcal{P}_n$: $\beta = \beta_2 \dots \beta_n = {}_n\beta \dots {}_2\beta$.* ("Combing")

Proof. Arrange the subwords $\in \mathcal{P}_n^j$ with ascending j by commuting them as described in lemma 2.1. The result is $\check{\beta}_2 \dots \check{\beta}_n$ with all $\check{\beta}_j \in \mathcal{P}_n^j$. Pushing $\check{\beta}_s$ to the end leaves it unchanged, because it has to pass only through those $\check{\beta}_j$ with $j > s$, thus $\check{\beta}_s = \beta_s$. The statement for descending order is proved correspondingly. □

Proof of theorem 1.

Given an arbitrary element $\beta \in \mathcal{B}_n$, we are looking for a conjugate $\hat{\beta} \in \mathcal{W}_n$ of β. Let $\Pi : \mathcal{B}_n \to S_n$ denote the homomorphism given by $\Pi(\sigma_i) = (i \quad i+1)$, then $\mathcal{P}_n = \ker \Pi$. Say $\Pi(\beta)$ consists of k cycles of the lengths $n_1, \dots, n_k \geq 1$ with $n = n_1 + \dots + n_k$. Since the cycle type determines conjugation classes in S_n, there exists a permutation $\tau \in S_n$ with

$$\begin{aligned} \tau^{-1}\Pi(\beta)\tau &= (n_1 \; n_1{-}1 \; \dots \; 1)(n_1{+}n_2 \; \dots \; n_1{+}1) \cdots (\sum_{i=1}^{k} n_i \; \dots \; \sum_{i=1}^{k-1} n_i{+}1) \\ &= \Pi(\pi_{n_1 \dots n_k}). \end{aligned} \tag{2.1}$$

Take any $\pi_\tau \in \mathcal{B}_n$ with $\Pi(\pi_\tau) = \tau$ and let $\tilde{\beta} := \pi_{n_1 \dots n_k}^{-1} \pi_\tau^{-1} \beta \, \pi_\tau$, then $\Pi(\tilde{\beta}) = 1_{S_n}$ thus $\tilde{\beta} \in \mathcal{P}_n$. Now let

$$\Delta(\alpha) := \pi_{n_1 \dots n_k} \alpha \, \pi_{n_1 \dots n_k}^{-1} \tag{2.2}$$

$$N := \{n_1, n_1{+}n_2, \dots, n_1{+}\dots{+}n_k\}, \quad \bar{N} := \{1, \dots, n\} \setminus N \tag{2.3}$$

$$F(\alpha) := \left\{ \, s \mid \text{string } s \text{ is not free in } \alpha^{-1}\tilde{\beta}\,\Delta(\alpha) \, \right\} \cap \bar{N} \tag{2.4}$$

then Δ is an automorphism of $\mathcal{P}_n$ and F is a set-valued function defined on $\mathcal{P}_n$ because for all $\alpha \in \mathcal{P}_n$ we have $\Pi\big(\alpha^{-1}\tilde{\beta}\,\Delta(\alpha)\big) = \Pi(\tilde{\beta}) = 1_{S_n}$.

We will now prove the existence of an $\alpha \in \mathcal{P}_n$ with $F(\alpha) = \emptyset$ inductively: Supposing there is an $\alpha_{(s-1)}$ with $\inf F\big(\alpha_{(s-1)}\big) > s-1$, we construct $\alpha_{(s)}$ with $\inf F\big(\alpha_{(s)}\big) > s$. Starting with $\alpha_{(0)} := 1_{\mathcal{P}_n}$, we thereby end up with $\alpha := \alpha_{(n)}$, having the property $\inf F\big(\alpha_{(n)}\big) > n$; this implies $F(\alpha) = \emptyset \quad (\inf \emptyset := +\infty)$.

1. $\inf F\big(\alpha_{(s-1)}\big) > s$: Simply take $\alpha_{(s)} := \alpha_{(s-1)}$.
2. $\inf F\big(\alpha_{(s-1)}\big) = s$: This can only happen for $s \in \bar{N}$. Now let

$$\delta \;:=\; (\alpha_{(s-1)})^{-1}\, \tilde{\beta}\, \Delta(\alpha_{(s-1)}) \tag{2.5}$$

$$\alpha_{(s)} \;:=\; \alpha_{(s-1)}\, ({}_s\delta) \tag{2.6}$$

$$\delta' \;:=\; (\alpha_{(s)})^{-1}\, \tilde{\beta}\, \Delta(\alpha_{(s)}) = ({}_s\delta)^{-1}\, \delta\, \Delta({}_s\delta). \tag{2.7}$$

String s is free in $({}_s\delta)^{-1}\delta$ (lemma 2.1). Furthermore the strings with numbers in $\{1,\ldots,s-1\} \cap \bar{N}$ are free in both δ (by induction hypothesis) and in ${}_s\delta$ (because ${}_s\delta \in \mathcal{P}_n^s$), hence they are free in $({}_s\delta)^{-1}\,\delta$, too.

For $s \in \bar{N}$ we have $\Delta(\mathcal{P}_n^s) \subset \mathcal{P}_n^{s+1}$, which can easily be verified, looking at the generators A_{is}; the three typical cases are shown in figure 2a–c. So the strings with numbers $\le s$ are also free in $\Delta({}_s\delta)$. Thus all strings with numbers in $\{1,\ldots,s\} \cap \bar{N}$ are free in δ' and (2.4) yields $\inf F\big(\alpha_{(s)}\big) > s$.

Now, having $F(\alpha) = \emptyset$ and $\tilde{\beta}$ as above, we can construct $\hat{\beta}$ as follows:

$$\hat{\beta} := \pi_{n_1\ldots n_k}\alpha^{-1}\tilde{\beta}\,\Delta(\alpha) = \big(\pi_\tau\,\Delta(\alpha)\big)^{-1}\beta\,\big(\pi_\tau\,\Delta(\alpha)\big). \tag{2.8}$$

With respect to the first expression, lemma 2.2 and the remark to lemma 2.1 together with $F(\alpha) = \emptyset$ show that $\hat{\beta}$ is of the form $\pi_{n_1\ldots n_k}\,\hat{\beta}_{n_1}\,\hat{\beta}_{n_1+n_2}\cdots\,\hat{\beta}_{n_1+\cdots+n_k}$. The last term of (2.8) makes clear that $\hat{\beta}$ is conjugate to β. $\square$

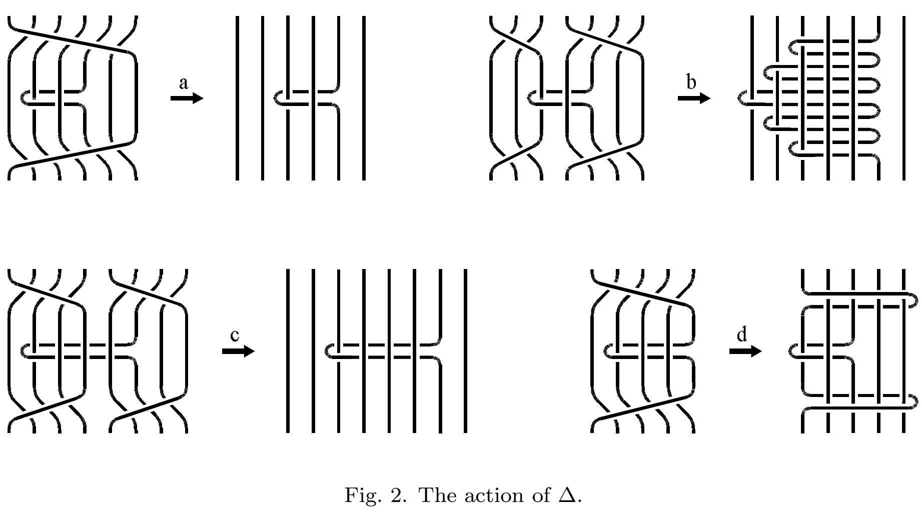

Fig. 2. The action of Δ.

Proof of corollary 1.

(a) and (b): Besides the results of [3] and [4] there is only to mention that closing a woven braid of type $(n_1, \ldots, n_k)$, placed parallel to the x-y-plane, around the x-axis and projecting it along the z-axis, yields a woven plat of type $(2n_1, \ldots, 2n_k)$. Hence every web of index n can be regarded as woven plat of index n (but not vice versa).

(c): If the bridge number of L is m, there exists a $2m$-plat representing L. We have to transform this plat into a woven plat without changing the number of strings. This is a task quite similar to the proof of theorem 1, so we will give only the crucial points here. Again, the first step is to order the strings by adjusting the permutation of the braid to $\Pi(\pi_{n_1 \ldots n_k})$. This can be done by using moves of type a and b in figure 3 at *both* ends of the plat.

In order to get rid of unwanted A_{ij}'s, we pushed them upwards and all around the closed braid. Passing $\pi_{n_1 \ldots n_k}$ increased the index j, with the result that the A's assembled at the highest available string of each component (those with numbers in N). In the case of plats we have to push A_{ij}'s with even $j \in \bar{N}$ upwards, they thereby have to pass $\pi_{n_1 n_2 \ldots n_k}$ and arrive as elements of $\mathcal{P}_n^{j+1}$ at the upper end. The A_{ij}'s with odd j have to be pushed down to the lower end of the plat. In both cases we can then apply the moves c and d of figure 3 or their mirror images repeatedly, in order to free the strings with numbers in $\bar{N}$ successively, just as before.

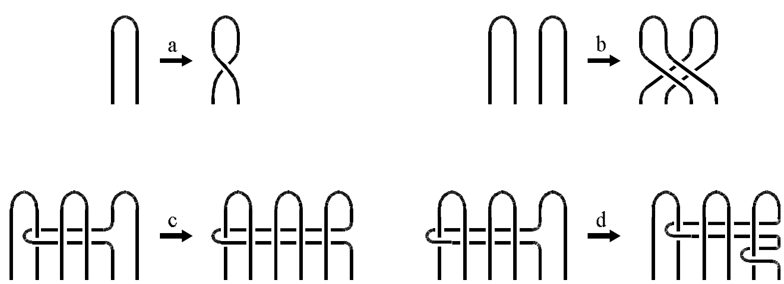

Fig. 3. Useful moves for plats, preserving the link type.

(d): Having more than two arcs per component or more than two colors makes finding an appropriate projection only easier, so we may assume that each embedded circle consists of exactly two arcs, colored black and gray. Choose $\tilde{l}$ to be a web equivalent to l, presented as required in definition 1. By parameter transformation we may arrange the coloring of $\tilde{l}$ in the way indicated in figure 4, identifying the black arcs with the 'moving' parts of the strings with numbers $\in N$. The obtained regular diagram has no monochrome crossings. $\square$

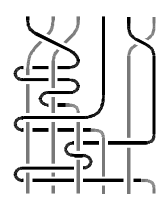

Fig. 4.

3. The algebraic Knot Problem for Webs

For a braid $\beta \in \mathcal{B}_n$ placed in the x-y-plane we will denote by $\mathrm{inv}(\beta)$ the braid obtained by π-rotating β on the z-axis. Observe that this rotation reverses the order of the braid word and sends σ_i to σ_{n+1-i}, so inv is an involution and an antihomomorphism.

Lemma 3.1. *If* $\omega \in \mathcal{W}_n^1$ *then* $\mathrm{inv}(\omega) \in \mathcal{W}_\mathrm{n}^1$.

Proof. Showing this algebraically is an easy task, so we will content ourselves with the following argumentation: The elements of $\mathcal{W}_n^1$ are characterized by the fact that they can be presented as braids with $n-1$ constant strings and one string moving arbitrarily from position 1 to position n; a property that is preserved under inv. □

Remark. The analogous statement for $\mathcal{W}_n^k$ is false when $k \geq 2$.

We will assume from now on that $k = 1$. Composing the $\mathcal{P}_n$-endomorphism ∇ of definition 2 and the $\mathcal{P}_n$-automorphism $\Delta(\alpha) = \pi_n \alpha\, \pi_n^{-1}$ of eq. (2.2) defines an endomorphism $\Phi_n := \nabla \circ \Delta : \mathcal{P}_n \to \mathcal{P}_n$ that will be useful for the characterization of $\mathcal{W}_n^1$ and $\mathcal{K}_n$.

Lemma 3.2. $\mathcal{W}_n^1 = \pi_n \ker(\Phi_n)$ *and* $\mathcal{K}_n = \{\kappa \in \mathcal{P}_n \mid \nabla(\kappa) = \Phi_n(\kappa)\}$.

Proof. From figure 2a,d we obtain (with an appropriate $\chi \in \mathcal{P}_n$)

$$\Delta(A_{ij}) = \begin{cases} A_{i+1,j+1} & \text{for } j < n \\ \chi^{-1} A_{1i}\, \chi & \text{for } j = n \end{cases} \quad \text{and thus } \Phi_n(A_{ij}) = \begin{cases} A_{ij} & \text{for } j < n \\ 1_{\mathcal{P}_n} & \text{for } j = n \end{cases}.$$

So $\ker(\Phi_n) = \mathcal{P}_n^n$, justifying the first statement. For any $\beta \in \mathcal{P}_n^n$ we get:[3]

$$\Phi_n\big(\beta\, \nabla(\beta) \cdots \nabla^{n-2}(\beta)\big) = \nabla(\beta) \cdots \nabla^{n-2}(\beta) = \nabla\big(\beta\, \nabla(\beta) \cdots \nabla^{n-2}(\beta)\big). \tag{3.1}$$

Conversely, by combing $\kappa \in \mathcal{P}_n$ in descending order $\kappa = {}_n\kappa \cdots {}_2\kappa$ we get

$$\begin{aligned} \nabla(\kappa) = \Phi_n(\kappa) \quad &\Rightarrow \quad \nabla({}_n\kappa) \cdots \nabla({}_3\kappa) = {}_{n-1}\kappa \cdots {}_2\kappa. \quad \Rightarrow \quad {}_{i-1}\kappa = \nabla({}_i\kappa) \\ &\Rightarrow \quad \kappa = {}_n\kappa\, \nabla({}_n\kappa)\, \nabla\big(\nabla({}_n\kappa)\big) \cdots \nabla^{n-2}({}_n\kappa) \in \mathcal{K}_n. \end{aligned} \tag{3.2}$$

In the second equation both sides are descendingly combed, so the second implication is just the uniqueness condition of lemma 2.1 for $2 < i \leq n$. □

Proof of theorem 2.

The simple relation $\sigma_n A_{in} \sigma_n^{-1} = A_{i,n+1}$ in $\mathcal{B}_{n+1}$ shows for $\beta \in \mathcal{P}_n^n$ that

$$\pi_n \beta\, \sigma_n^{-1} = \pi_{n+1}\, \sigma_n^{-2}\, \beta^{\uparrow} \quad \text{and} \quad \pi_n \beta\, \sigma_n = \pi_{n+1}\, \sigma_n^{-2}\, \beta^{\uparrow} \sigma_n^2 \tag{3.3}$$

where $\beta^{\uparrow} \in \mathcal{P}_{n+1}^{n+1}$ is obtained from β by augmenting the second index of all A_{ij}'s by one. Since $\sigma_n^2 = A_{n,n+1}$, we see that a type II$^+$ move yields a woven braid with incremented index.

[3] According to eq. (1.3) we have $\nabla(\mathcal{P}_n^j) = \mathcal{P}_n^{j-1}$ for $j \geq 3$ and $\nabla^{n-1}(\beta) = 1_{\mathcal{P}_n}$.

We have for $\omega \in \mathcal{W}_n^1$ and $\kappa \in \mathcal{P}_n$ the following consequence of lemma 3.2:

$$\begin{aligned}
\kappa^{-1}\omega\kappa \in \mathcal{W}_n^1 \quad &\Leftrightarrow \quad \pi_n^{-1}\kappa^{-1}\omega\kappa = \Delta^{-1}(\kappa^{-1})\pi_n^{-1}\omega\kappa \in \ker\Phi_n \\
&\Leftrightarrow \quad \Phi_n\big(\Delta^{-1}(\kappa^{-1})\big)\Phi_n(\pi_n^{-1}\omega)\Phi_n(\kappa) = \nabla(\kappa^{-1})\Phi_n(\kappa) = 1_{\mathcal{P}_n} \\
&\Leftrightarrow \quad \nabla(\kappa) = \Phi_n(\kappa) \quad \Leftrightarrow \quad \kappa \in \mathcal{K}_n
\end{aligned} \tag{3.4}$$

The "$\Leftarrow$"-direction states that the result of a type I move is again a woven braid. Conversely we get that $\mathcal{K}_n$ is a subgroup of $\mathcal{P}_n$ because for any $\kappa_1, \kappa_2 \in \mathcal{K}_n$ the woven braids $\kappa_1^{-1}\omega\kappa_1$ and $\kappa_2^{-1}\omega\kappa_2$ are conjugate by $\kappa_1^{-1}\kappa_2 \in \mathcal{P}_n$ and the "$\Rightarrow$"-direction thus implies $\kappa_1^{-1}\kappa_2 \in \mathcal{K}_n$.

Let $\omega, \gamma^{-1}\omega\gamma \in \mathcal{W}_n^1$ with $\gamma \in \mathcal{B}_n$. Then $\Pi(\gamma^{-1}\omega\gamma) = \Pi(\omega) \Rightarrow \big[\Pi(\gamma), \Pi(\omega)\big] = 1_{S_n}$; but since $\Pi(\omega) = (n\ \ldots\ 1)$ consists of exactly one cycle, this is only possible if $\Pi(\gamma) = \big(\Pi(\omega)\big)^i$ with some integer i. Thus we have a pure braid $\tilde{\gamma} := \omega^{-i}\gamma \in \mathcal{P}_n$ giving the same conjugation result: $\gamma^{-1}\omega\gamma = \tilde{\gamma}^{-1}\omega\tilde{\gamma}$. According to (3.4) $\tilde{\gamma} \in \mathcal{K}_n$, showing that any two conjugate woven braids are related via a single type I move.

Markov's theorem states that the closures of two braids represent the same link if and only if one braid can be transformed into the other by conjugation and stabilization-moves. So it only remains to show, that for any $\beta \in \mathcal{B}_n$ and $\beta' = \beta\sigma_n^{\pm 1} \in \mathcal{B}_{n+1}$ there are woven braids $\omega \in \mathcal{W}_n^1$ and $\omega' \in \mathcal{W}_{n+1}^1$ that are related by a type II$^{\pm}$ move and are contained in the conjugation classes of β and β', respectively.

Similar to the proof of theorem 1 we construct a $\gamma \in \mathcal{B}_n$ with $\gamma^{-1}\mathrm{inv}(\beta)\gamma \in \mathcal{W}_n^1$. According to equations (2.8) and (2.6) we obtain $\gamma = \pi_\tau\Delta(\alpha)$ where α contains only the generators A_{ij} with $j < n$. Therefore the action of Δ on α is described by figure 2a, and thus $\Delta(\alpha) \in \langle (A_{ij})_{2\le i<j\le n}\rangle$.

The requirement $\tau^{-1}\Pi(\mathrm{inv}(\beta))\tau = \Pi(\pi_n)$ of equation (2.1) determines the permutation τ not uniquely: any other $\tilde{\tau} := (\Pi(\pi_n))^i\,\tau$ works as well. Since $\Pi(\pi_n)$ is a cycle of length n, there is an integer i such that $\tilde{\tau}(1) = 1$ and we can choose $\pi_\tau \in \langle (\sigma_i)_{2\le i<n}\rangle$.

We have therefore found a γ that can be written as word of $\mathcal{B}_n$ without using σ_1. So in $\tilde{\gamma} := \mathrm{inv}(\gamma)$ no σ_{n-1}'s appear and we have $[\tilde{\gamma}, \sigma_n] = 1_{\mathcal{B}_{n+1}}$ in $\mathcal{B}_{n+1}$. We can now take

$$\begin{aligned}
\omega \quad &:= \quad \mathrm{inv}(\gamma^{-1}\mathrm{inv}(\beta)\gamma) = \tilde{\gamma}\beta\tilde{\gamma}^{-1} \quad \text{and} \\
\omega' \quad &:= \quad \tilde{\gamma}\beta'\tilde{\gamma}^{-1} = \tilde{\gamma}\beta\sigma_n^{\pm 1}\tilde{\gamma}^{-1} = \tilde{\gamma}\beta\tilde{\gamma}^{-1}\sigma_n^{\pm 1} = \omega\sigma_n^{\pm 1}.
\end{aligned} \tag{3.5}$$

By lemma 3.1 ω is a woven braid and ω' is obviously the result of a type II$^+$ move applied on ω. This completes the proof of theorem 2. □

Remark. The premise $k = 1$ was very helpful at several points of this proof and things are getting much more complicated for links. For $k > 2$ conjugations with non-pure braids are inevitable because one must have moves that exchange components. It is likely, however, that – with some additional moves – theorem 2 can be extended to the case of colored links.

4. Tables

4.1. Link-tabulation with woven braids

Due to corollary 1 links can be tabulated by giving a woven braid ω_L for each equivalence class L, with either the closure of ω_L or the plat formed out of ω_L is contained in L. Depending on the construction we will call $(\omega_L)_{L\in\mathcal{L}}$ a web- or a plat-table for the family $\mathcal{L}$ of links. We have the nice side effect that orientation and labelling of the components are provided automatically, if we agree upon the convention that all strings of the closure and all odd-numbered strings of the plat shall be oriented in reading direction of the braid word and that components shall be labelled increasingly with respect to their position.

In comparison with tables based on diagrams we lose the information about the crossing numbers and alternation type.[4] On the other side, web-/plat-tables can implicitly convey the braid-/bridge-indices and in some cases they reveal strong symmetries. A knot is called strongly invertible / strongly negative amphicheiral if there is an orientation preserving / reversing self-homeomorphism of S^3 that is an involution fixing the knot setwise and reversing the knot orientation. A woven braid ω is called *symmetric* if $\mathrm{inv}(\omega) = \omega$ and *antisymmetric* if $\mathrm{inv}(\omega) = \mathrm{mirr}(\omega)$, with the automorphism mirr given by $\mathrm{mirr}(\sigma_i) := \sigma_i^{-1}$. It is a simple consequence that a web or woven plat build out of a symmetric / antisymmetric woven braid is strongly invertible / strongly negative amphicheiral.

4.2. Notation

For woven braids we have a trivial solution of the word problem of $\mathcal{B}_n$: Let the woven braid be presented in the form required in definition 1, expand all A_{ij} according to (1.1) and then reduce until no more subwords of type $\sigma_i^{\pm 1}\sigma_i^{\mp 1}$ occur. The result, which will be called *tight* word, is unique because the $\mathcal{P}_n^i$ are free. In general, tight words are not minimal in length; $(\sigma_2\sigma_1^{-1})^2$ for example yields the same braid as the tight word $\sigma_1^{-1}\sigma_2^{-2}\sigma_1^2\sigma_2$.

Tight words could in principle be written down by listing the indices and exponents of the σ_i's, but we will use a much more efficient notation for knots making use of the fact that in tight words of $\mathcal{W}_n^1$ every σ_i involves the string starting at position 1.

Lemma 4.1. *For any tight word $w = \sigma_{i_1}^{e_1}\ldots\sigma_{i_l}^{e_l}$ of $\mathcal{W}_n^1$ with $1 \le i_t < n$ and $e_t = \pm 1$ the l-tuple $(e_1, c_1, c_2, \ldots, c_{l-1})$ determines w uniquely, if we take*

$$c_t := \left\{ \begin{array}{ll} 0 & \textit{if } i_t \neq i_{t+1} \textit{ and } e_t = e_{t+1} \\ 1 & \textit{if } i_t \neq i_{t+1} \textit{ and } e_t \neq e_{t+1} \\ 2 & \textit{if } i_t = i_{t+1} \end{array} \right\} \quad \textit{for } 1 \le t \le l-1. \tag{4.1}$$

[4] For instance, one can show that any web representing the figure-eight knot must have at least six crossings and cannot be alternating.

Proof. The indices and exponents can be recovered recursively:

$$\begin{array}{rclrcl} d_1 & := & 1, & d_{t+1} & := & (-1)^{\frac{1}{2}c_t(c_t-1)}d_t \\ i_1 & := & 1, & i_{t+1} & := & i_t + \bigl(1 - \frac{1}{2}c_t(c_t-1)\bigr)d_t \\ & & & e_{t+1} & := & (-1)^{c_t}e_t. \end{array} \tag{4.2}$$

This statement is obviously correct for the e_t. We call the string moving from first position in the braid to position n the *main* string. Observing that $\frac{1}{2}c_t(c_t-1) = 1$ for $c_t = 2$ and $= 0$ otherwise, we realize that d_t describes the direction of the main string at the crossing $\sigma_{i_t}^{e_t}$. Indeed $d_1 = 1$ (towards higher indices) at the beginning of the braid, and a sign change takes place whenever two consecutive σ's are at the same level (at U-turns). The index i_t is increased or decreased whenever $c_t \neq 2$, depending on the current direction d_t. □

In web-tables we can even strip off leading and tailing zeros and use the tuple $(e_1, c_u, \ldots, c_v)$ instead, where u and v are min and max of $\{j \mid c_j \neq 0\}$, respectively. The reason is that if the tuples of woven braids differ only by the numbers of their leading and tailing zeros, then the braids are related by type $\mathrm{II}^{\pm}$ and $\mathrm{inv} \circ \mathrm{II}^{\pm} \circ \mathrm{inv}$ moves and their closures are of the same knot type. The tuple $(e_1, c_u, \ldots, c_v)$ shall therefore refer to the one with lowest string index, which is uniquely determined.

Remark.

(a) We have hereby reduced braid words of length l with $2(n-1)$ different letters to words of length $\leq l$ using only three letters.

(b) If $(e_1, c_u, \ldots, c_v)$ is the tuple describing ω then $\mathrm{mirr}(\omega)$ is given by the tuple $(-e_1, c_u, \ldots, c_v)$ and $\mathrm{inv}(\omega)$ by $((-1)^{c_u+\cdots+c_v}e_1, c_v, \ldots, c_u)$.

(c) An analogous notation for links can be given as well by adding the letter “3”, signalling the passage to the next component.

4.3. Minimal webs

Definition 3. We call the closure of a woven braid *minimal*, if and only if

1. the number of strings equals its braid index, and
2. its length (i.e. the length of its tight braid word) is minimal amongst all other webs satisfying condition 1 and representing the same oriented (colored) link.

Remark. Minimal webs have not necessarily the minimal length of all equivalent webs: Computer calculations revealed that $\sigma_1\sigma_2^{-2}\sigma_1^2\sigma_2\sigma_3^{-2}\sigma_2\sigma_1^{-2}\sigma_2^{-2}\sigma_1^2\sigma_2\sigma_3 \in \mathcal{W}_4^1$ gives a minimal web representing the knot 9_{34}, but closing the shorter woven braid $\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1^{-2}\sigma_2\sigma_3^{-1}\sigma_4^{-2}\sigma_3\sigma_2^{-2}\sigma_3\sigma_4 \in \mathcal{W}_5^1$ yields 9_{34} as well.

Table 1 lists it all minimal webs for the prime knots with at most nine crossings. The given woven braids are Markov-equivalent to the braids listed in [6].[5]

[5]This condition determines the orientations and which knot out of chiral pairs is chosen. Take into account that in [6] the mirror images of the σ_i's are taken as generators of $\mathcal{B}_n$!

Table 1. Minimal webs for prime knots with crossing number ≤ 9.

AB	Webs	b	l	n	s
3_1	$-\mathit{5}^{\mathrm{s}}$	2	3	1	r
4_1	$\pm 15^{\mathrm{a}}$	3	6	2	f
5_1	$-\mathit{53}^{\mathrm{s}}$	2	5	1	r
5_2	-31	3	6	2	r
6_1	39	4	9	2	r
6_2	$-143, -419$	3	8	4	r
6_3	$\pm \mathit{100}^{\mathrm{a}}$	3	6	2	f
7_1	$-\mathit{485}^{\mathrm{s}}$	2	7	1	r
7_2	-91	4	9	2	r
7_3	-319	3	8	2	r
7_4	1794^{s}	4	9	1	r
7_5	$-287, -851$	3	8	4	r
7_6	-99	4	9	2	r
7_7	$-3741^{\mathrm{s}}, 16986^{\mathrm{s}}$	4	11	2	r
8_1	111	5	12	2	r
8_2	$-1295, -3779$	3	10	4	r
8_3	$\pm 115^{\mathrm{a}}$	5	12	2	f
8_4	$-359, -1067$	4	11	4	r
8_5	-3875	3	10	2	r
8_6	$-395, -1175, -3515$	4	11	6	r
8_7	$\mathit{916}$	3	8	2	r
8_8	-280	4	9	2	r
8_9	$\pm 1439^{\mathrm{a}}, \pm 4031$	3	10	6	f
8_{10}	$-\mathit{911}$	3	8	2	r
8_{11}	375	4	11	2	r
8_{12}	$\pm 135^{\mathrm{a}}$	5	12	2	f
8_{13}	1814	4	9	2	r
8_{14}	$7629, -16970$	4	11	4	r
8_{15}	$-2799, -3701, -7699$	4	11	6	r
8_{16}	-12521^{s}	3	10	1	r
8_{17}	$1275^{\mathrm{an}}, -2715^{\mathrm{n}}, -3772^{\mathrm{n}}, 12083^{\mathrm{an}}$	3	10	4	i
8_{18}	$\pm 7708^{\mathrm{a}}$	3	10	2	f
8_{19}	-365	3	8	2	r
8_{20}	371	3	8	2	r
8_{21}	-905	3	8	2	r
9_1	$-\mathit{4373}^{\mathrm{s}}$	2	9	1	r
9_2	-271	5	12	2	r
9_3	-2911	3	10	2	r
9_4	-955	4	11	2	r
9_5	14106	5	12	2	r
9_6	$-2591, -7667$	3	10	4	r
9_7	$-827, -2471, -7403$	4	11	6	r
9_8	-291	5	12	2	r

AB	Webs	b	l	n	s
9_9	$-2879, -8627$	3	10	4	r
9_{10}	16482^{s}	4	11	1	r
9_{11}	-963	4	11	2	r
9_{12}	-295	5	12	2	r
9_{13}	$16162, 16194$	4	11	4	r
9_{14}	127794	5	14	2	r
9_{15}	-279	5	12	2	r
9_{16}	-7763	3	10	2	r
9_{17}	$33981^{\mathrm{s}}, -153066^{\mathrm{s}}$	4	13	2	r
9_{18}	-859	4	11	2	r
9_{19}	$31605, -128766$	5	14	4	r
9_{20}	$-899, -2687$	4	11	4	r
9_{21}	14322	5	12	2	r
9_{22}	$33693, -152885$	4	13	4	r
9_{23}	$-20858^{\mathrm{s}}, 22882^{\mathrm{s}}$	4	11	2	r
9_{24}	$1235, 3695$	4	11	4	r
9_{25}	$-25407, 29847, 33279, -32897, -69055$	5	14	10	r
9_{26}	$-33677, 152890, 152922$	4	13	6	r
9_{27}	-915	4	11	2	r
9_{28}	$7576, -7589, -23104$	4	11	6	r
9_{29}	-691971	4	15	2	r
9_{30}	-8169	4	11	2	r
9_{31}	$-\mathit{2354}^{\mathrm{s}}$	4	9	1	r
9_{32}	$-3771^{\mathrm{n}}, -108717^{\mathrm{n}}$	4	13	2	n
9_{33}	$23739^{\mathrm{n}}, -69148^{\mathrm{n}}, 101459^{\mathrm{n}}$	4	13	3	n
9_{34}	$-1362201, -1976361, 12383186$	4	17	6	r
9_{35}	$436350, 3427966$	5	16	4	r
9_{36}	-8051	4	11	2	r
9_{37}	$94809^{\mathrm{s}}, -168523, 218815, -3474018^{\mathrm{s}}$	5	16	6	r
9_{38}	145570	4	13	2	r
9_{39}	$500919, 3480870$	5	16	4	r
9_{40}	-5915517	4	17	2	r
9_{41}	$12114242, -15094041, -35669036$	5	18	6	r
9_{42}	-3191	4	11	2	r
9_{43}	$-3197, 3255$	4	11	4	r
9_{44}	$-8057, 8115$	4	11	4	r
9_{45}	-3309	4	11	2	r
9_{46}	9627	4	13	2	r
9_{47}	-9573	4	13	2	r
9_{48}	145522	4	13	2	r
9_{49}	$-62113, 147666$	4	13	4	r

Minimality has been checked by calculating HOMFLY- and Kauffman-polynomials for all shorter woven braids; braid indices can be calculated using the MFW-inequality (see [6]).

The column "AB" lists the Alexander-Briggs notations, "b","l" and "n" give the

braid indices, the lengths and the number of minimal webs for the corresponding knot, respectively. The column 's', taken out of [7], contains information about knot symmetries in Conway's notation:

	amphicheiral	non-amphicheiral
invertible	f	r
non-invertible	i	n

The tuples $(e_1, c_u, \ldots, c_v)$ of the minimal webs are listed in the second column, encoded as integers $e_1(c_u - 1 + 2c_{u+1} + 6c_{u+2} + \cdots + 2 \cdot 3^{v-u-1}c_v)$. If the woven braids ω and $\mathrm{inv}(\omega)$ represent the same oriented knot, we list only the absolutely smaller of the two corresponding integers. So most of the integers represent two webs; exceptions are marked by the following superscripts:

n: The knot is non-invertible.

s: ω is symmetric, showing that the knot is strongly invertible.

a: ω is antisymmetric, showing the strong negative amphicheirality.

An entry of the form $\pm j$ indicates that j and $-j$ appear (i.e. amphicheirality). Integers in italics represent webs with alternating tight words.

In table 1 we get, on average, 2.9 minimal webs per knot type, having an average tight word length of 11.5. Symmetric woven braids occur for twelve knots and all seven amphicheiral knots in the table have antisymmetric minimal woven braids. The author also found antisymmetric web-representatives for the amphicheiral ten-crossing knots, being minimal except in the case of 10_{81} (a list of minimal webs for all ten-crossing knots is available on request). This demonstrates that all amphicheiral knots with at most ten crossings are strongly negative amphicheiral.

It will be the task of a subsequent article to show that any strongly negative amphicheiral knot can be presented as closure of some antisymmetric woven braid.

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